Published by Institute of Physics Publishing for SISSA/ISAS

Received: *February 22, 2005*
Revised: *March 1, 2005*
Accepted: *March 1, 2005*

Can one tell Einstein's unimodular theory from Einstein's general relativity?

Enrique Álvarez

Instituto de Física Teórica UAM/CSIC, C-XVI
and Departamento de Física Teórica, C-XI,
Universidad Autónoma de Madrid
E-28049-Madrid, Spain
E-mail: Enrique.Alvarez@uam.es

Abstract: The so called unimodular theory of gravitation is compared with general relativity in the quadratic (Fierz-Pauli) regime, using a quite broad framework, and it is argued that quantum effects allow in principle to discriminate between both theories.

Keywords: Bosonic Strings, Space-Time Symmetries, Classical Theories of Gravity, Models of Quantum Gravity.

Contents

1. Introduction

Although it does not seem to be generally known (see, however, a footnote in [1]), four years after writing down the equations of general relativity, Einstein [2] also proposed a different set of equations, what have subsequently been dubbed as corresponding to *unimodular gravity*. The (english translation of the) title of Einstein's paper is: *Do gravitational fields play an essential part in the structure of the elementary particles of matter?*, and its purpose was to obtain an alternative to Mie's theory on the stability of the electron, and as such, it was unsuccessful. Quoting Einstein himself: *... Thus the problem of the constitution of the elementary quanta cannot yet be solved on the immediate basis of the given field equations.*

But on the way, he realized already in 1919 that the unimodular theory is equivalent to general relativity, with the cosmological constant appearing as an integration constant. Let us quickly recall how this comes about.

The posited equations of motion are the tracefree part of Einstein's general relativity ones (written in dimension n):

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right) \tag{1.1}$$

(with $\kappa^2 \equiv 8\pi G$). It seems that there is less information here, because the trace has been left out, but this is deceptive: the contracted Bianchi identities guarantee that

$$\nabla^\nu R_{\mu\nu} = \frac{1}{2} \nabla_\mu R\,. \tag{1.2}$$

Applying this to the equations in the set (1.1) we get (assuming covariant conservation of the energy-momentum tensor)

$$\frac{n-2}{2}\nabla_\mu R = -\frac{\kappa^2}{n}\nabla_\mu T \tag{1.3}$$

which integrates to

$$\frac{n-2}{2d}R + \frac{2\kappa^2}{d}T = \text{constant} \equiv -\lambda \tag{1.4}$$

and plugging this fact into the equation (3.27) yields precisely Einstein's general relativity equations, with arbitrary cosmological constant, λ:

$$R_{\mu\nu} - \frac{1}{2}(R+2\lambda)g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{1.5}$$

(Signs are chosen in such a way that with the Landau-Lifshitz spacelike conventions de Sitter space corresponds to a positive cosmological constant).

This result remains true when including higher order in curvature corrections, in the following sense.

If the equations of motion can be derived from a covariant action principle, schematically

$$S \equiv \int d(vol)L(g_{\mu\nu}, R_{\mu\nu\rho\sigma}) \tag{1.6}$$

there is a generalization of the Bianchi identity which ensures that

$$\nabla_\mu \frac{\delta S}{\delta g_{\mu\nu}} \equiv 0\,. \tag{1.7}$$

This Bianchi identity, in turn, allows the trace $g_{\alpha\beta}\frac{\delta S}{\delta g_{\alpha\beta}}$ to be recovered as an integration constant from the tracefree component of the equations of motion.

Some previous work on this theory is [3, 4, 5, 6, 1, 7, 8, 9].

In the paper by van der Bij, van Dam and Ng, in particular, it is proven that the famous argument, coming from the need to represent trivially the translations of the little group of massless particles, $ISO(2)$ that leads to abelian $U(1)$ gauge invariance in the spin one case, also leads to the unimodular theory in the spin two case. This means that it is enough for this purpose (namely, to represent translations trivially), to impose the gauge symmetry at the linear level

$$\delta h_{\alpha\beta} = \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha \tag{1.8}$$

with

$$\partial_\alpha \xi^\alpha = 0 \tag{1.9}$$

instead of the Fierz-Pauli [10] full symmetry without the transversality condition.

The purpose of the present work is the quite modest one of answering the question in the title, as to whether it is at all possible to discrimitate between the two Einstein's theories. Al the classical level, the equations are identical, so it seems that attention should be focused in quantum effects.

2. The action principle

Einstein never talks about an action principle in his paper. And this for a good reason, given the fact that he was only interested in the equations of motion. But if we are interested in quantum effects, this is not enough, and we have to invoke a particular lagrangian. There are several ways to do this, and each one of them defines *a priori* a different quantum theory. Some of the alternatives have been discussed in the references, in particular in [1] and in [4].

The full action comprising the metric and the matter fields, ϕ_i, will be represented as

$$S = S_{\rm grav} + S_{\rm matt} \tag{2.1}$$

with

$$S_{\rm grav} \equiv \int \sqrt{|g|} d^n x L(g_{\alpha\beta}, R_{\alpha\beta\gamma\delta}) \tag{2.2}$$

and

$$S_{\rm matt} \equiv \int \sqrt{|g|} d^n x L(g_{\alpha\beta}, \phi_i) \,. \tag{2.3}$$

The allowed variations , which we be denoted as $\delta^t g_{\alpha\beta}$ are constrained by

$$\delta^t \sqrt{|g|} = 0 \tag{2.4}$$

that is

$$g^{\alpha\beta} \delta^t g_{\alpha\beta} = 0 \tag{2.5}$$

which can be expressed in terms of an unconstrained variation as

$$\delta^t g_{\alpha\beta} = \delta g_{\alpha\beta} - \frac{1}{n} g_{\alpha\beta} g^{\mu\nu} \delta g_{\mu\nu} \,. \tag{2.6}$$

It is clear that any variation can be expressed as

$$\delta S = \int d^n x \frac{\delta S}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} = \int d^n x \frac{\delta S}{\delta g_{\alpha\beta}} \left(\delta^t g_{\alpha\beta} + \frac{1}{n} g_{\alpha\beta} g^{\mu\nu} \delta g_{\mu\nu} \right) \tag{2.7}$$

so that the restricted variation is just the trace-free part of the unconstrained variation:

$$\frac{\delta S}{\delta^t g_{\alpha\beta}} = \frac{\delta S}{\delta g_{\alpha\beta}} - \frac{1}{n} g^{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} g_{\alpha\beta} \tag{2.8}$$

and this variational principle indeed yields Einstein's unimodular field equations as written down in the introduction when the Hilbert lagrangian is used as an starting point.

Were we to put forward a stronger claim, namely that all physics is invariant under APD only, which would then the basic symmetry principle which takes the place of diffeomorphism invariance (DI), then the energy momentum tensor would not be fully covariantly conserved. The assumed symmetry only guarantees that

$$\int \sqrt{|g|} d^n x (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) \frac{\delta L_{\rm matt}}{\delta g_{\alpha\beta}} = 0 \tag{2.9}$$

whenever the vector field ξ satisfies the transversality condition

$$\nabla_\alpha \xi^\alpha = 0 \tag{2.10}$$

which can be locally integrated in terms of an $(n-2)$-differential form, Ω, to

$$\xi^\alpha = \frac{1}{\sqrt{|g|}} \delta^{\alpha\beta\mu_1\ldots\mu_{n-2}} \nabla_\beta \Omega_{\mu_1\ldots\mu_{n-2}} \,. \tag{2.11}$$

The conservation law for $\Theta_{\alpha\beta} \equiv \frac{2}{\sqrt{|g|}} \frac{\delta S_{\rm matt}}{\delta g_{\alpha\beta}}$ is then

$$\int \sqrt{|g|} d^n x \left(\nabla^\alpha \frac{1}{\sqrt{|g|}} \delta^{\beta\delta\mu_1\ldots\mu_{n-2}} \nabla_\delta \Omega_{\mu_1\ldots\mu_{n-2}} + \nabla^\beta \frac{1}{\sqrt{|g|}} \delta^{\alpha\delta\mu_1\ldots\mu_{n-2}} \nabla_\delta \Omega_{\mu_1\ldots\mu_{n-2}} \right) \Theta_{\alpha\beta} \tag{2.12}$$

that is

$$\nabla_\mu \Theta^{\mu\nu} = \nabla^\nu \phi \tag{2.13}$$

for some scalar ϕ.

This is consistent with our previous finding that

$$\Theta_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{n} T g_{\alpha\beta} \tag{2.14}$$

in such a way that indeed

$$\nabla_\alpha \Theta^{\alpha\beta} = \frac{1}{n} \nabla^\beta T \,. \tag{2.15}$$

3. Quadratic analysis

Let us start with the well-known analysis which leads eventually to the Fierz-Pauli lagrangian for a free massless spin two particle (cf. [11, 12]). A simple road is as follows: the quadratic part of the lagrangian is the inverse of the propagator, and the propagator is related to the possible polarizations. There are five of those in the massive spin two case, which can be represented as $\epsilon^A_{\mu\nu}$ $A = 1\ldots 5$, with

$$\begin{aligned} \epsilon^A_{\mu\nu} &= \epsilon^A_{\nu\mu} \\ k^\mu \epsilon^A_{\mu\nu} &= 0 \\ \eta^{\mu\nu} \epsilon^A_{\mu\nu} &= 0 \end{aligned} \tag{3.1}$$

We can expand the momentum space[1] propagator in terms of the basic tensors k^μ and the off-shell transverse projection operator $\eta^T_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ as

$$\begin{aligned} D_{\mu\nu\lambda\sigma} \equiv \sum_A \epsilon^A_{\mu\nu}\epsilon^A_{\lambda\sigma} = {} & c_1 \eta^T_{\mu\nu}\eta^T_{\lambda\sigma} + c_2 \eta^T_{\mu\nu} k_\lambda k_\sigma + k_\mu k_\nu \eta^T_{\lambda\sigma} + c_3 (\eta^T_{\mu\lambda}\eta^T_{\nu\sigma} + \eta^T_{\mu\sigma}\eta^T_{\nu\lambda}) + \\ & + c_4 (k_\mu k_\sigma \eta^T_{\nu\lambda} + k_\mu k_\lambda \eta^T_{\nu\sigma} + k_\nu k_\sigma \eta^T_{\mu\lambda} + k_\nu k_\lambda \eta^T_{\mu\sigma} + c_5 k_\mu k_\nu k_\lambda k_\sigma \,. \end{aligned} \tag{3.2}$$

[1]Both position and momentum space notation will be used for convenience. Although most formulas will be written in arbitrary dimension, most of the polarization reasoning is implicitly four-dimensional.

Imposing off-shell transversality and tracelessness we get uniquely

$$D_{\mu\nu\lambda\sigma} = c_1 \left(\eta^T_{\mu\nu}\eta^T_{\lambda\sigma} - \frac{3}{2}\left(\eta^T_{\mu\lambda}\eta^T_{\nu\sigma} + \eta^T_{\mu\sigma}\eta^T_{\nu\lambda}\right)\right). \tag{3.3}$$

Acting on conserved currents, we can drop the superscript T.

In order to find the lagrangian, we have to compute the propagator by imposing transversality on shell only. Otherwise there are unwanted degeneracies. This amounts to change the projector in (3.3) $\eta^T_{\mu\nu}$ for a quantity $\eta^{\rm TOS}_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$, which behaves as a projector on shell only:

$$\begin{aligned}
\eta^{\rm TOS}_{\mu\nu} k^\nu &= k_\mu \frac{m^2 - k^2}{m^2} \\
\eta^{\rm TOS}_{\mu\nu}\eta^{\mu\nu} &= 3 + \frac{m^2 - k^2}{m^2} \\
\eta^{\rm TOS}_{\mu\nu}(\eta^{\rm TOS})^{\nu\rho} &= \eta^{\rm TOS\,\rho}_{\mu} + \frac{k^2 - m^2}{m^4} k_\mu k^\rho
\end{aligned} \tag{3.4}$$

What remains is

$$D^m_{\mu\nu\lambda\sigma} = c_1 \left(\eta^{\rm TOS}_{\mu\nu}\eta^{\rm TOS}_{\lambda\sigma} - \frac{3}{2}\left(\eta^{\rm TOS}_{\mu\lambda}\eta^{\rm TOS}_{\nu\sigma} + \eta^{\rm TOS}_{\mu\sigma}\eta^{\rm TOS}_{\nu\lambda}\right)\right) . \tag{3.5}$$

The lagrangian is then found by computing the inverse.

$$(K^m)_{\mu\nu\alpha\beta}(D^m)^{\alpha\beta}{}_{\lambda\delta} = \frac{1}{2}(\eta_{\mu\lambda}\eta_{\nu\delta} + \eta_{\mu\delta}\eta_{\lambda\nu}) \tag{3.6}$$

The conventional normalization corresponds to

$$c_1 = -\frac{4}{3}\frac{1}{k^2 - m^2} \tag{3.7}$$

and yields

$$\begin{aligned}
(K^m)_{\mu\nu\rho\sigma} = &\frac{k^2 - m^2}{8}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - 2\eta_{\mu\nu}\eta_{\rho\sigma}) - \\
&-\frac{1}{8}(k_\mu k_\rho \eta_{\nu\sigma} + k_\nu k_\sigma \eta_{\mu\rho} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} - 2k_\mu k_\nu \eta_{\rho\sigma} - 2k_\rho k_\sigma \eta_{\mu\nu})
\end{aligned} \tag{3.8}$$

which corresponds to the Fierz-Pauli lagrangian

$$L_{FP} = \frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} - \frac{1}{2}\partial_\mu h^{\nu\rho}\partial^\nu h_{\mu\rho} + \frac{1}{2}\partial_\mu h \partial^\sigma h_{\sigma\mu} - \frac{1}{4}\partial_\mu h \partial^\mu h - \frac{m^2}{4}\left(h_{\alpha\beta}h^{\alpha\beta} - h^2\right) \tag{3.9}$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$.

It follows that

$$k^\nu K^m_{\mu\nu\rho\sigma} h^{\rho\sigma} = -2m^2(k^\rho h_{\rho\mu} - k_\mu h) \tag{3.10}$$

so that necessarily,

$$k^2 h = k_\rho k_\sigma h^{\rho\sigma} . \tag{3.11}$$

The trace gives:

$$\eta^{\mu\nu} K^m_{\mu\nu\rho\sigma} h^{\rho\sigma} = -2(1 - n)m^2 h \tag{3.12}$$

which in turn implies that

$$h = k_\mu k_\nu h^{\mu\nu} = 0 \tag{3.13}$$

and using (3.10),

$$k^\mu h_{\mu\nu} = 0 \tag{3.14}$$

so that the field obeys the Klein-Gordon equation

$$(\Box + m^2) h_{\mu\nu} = 0\,. \tag{3.15}$$

It can be shown ([13]) that this particular mass term is the only one which is compatible with unitarity.

3.1 The massless limit.

The massless limit is singular. Three polarizations can be written as

$$k_\mu u_\nu + k_\nu u_\mu \tag{3.16}$$

with $k.u = 0$. Namely, in an obvious notation, ($e_{(a)} \equiv \partial_a$, etc)

$$\begin{aligned} & k \otimes k \\ & k \otimes e_{(1)} + e_{(1)} \otimes k \\ & k \otimes e_{(2)} + e_{(2)} \otimes k \end{aligned} \tag{3.17}$$

The remaining two are

$$\begin{aligned} \epsilon_1 &\equiv e_{(1)} \otimes e_{(2)} + e_{(2)} \otimes e_{(1)} \\ \epsilon_2 &\equiv e_{(1)} \otimes e_{(1)} - e_{(2)} \otimes e_{(2)} \end{aligned} \tag{3.18}$$

and under the little group, they transform into the other three (cf. [7]).

This means that exactly the same type of reasoning that gives rise to the abelian gauge invariance yields the unimodular theory of Einstein, which is invariant under area preserving diffs only:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{3.19}$$

with

$$\partial_\mu \xi^\mu = 0\,. \tag{3.20}$$

Once we implement this symmetry (with or without the unimodularity condition (3.20)), then there is a gauge in which the massless Fierz-Pauli propagator is defined up to a constant as:

$$D^{GF}_{\mu\nu\rho\sigma} = c_2(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})\,. \tag{3.21}$$

And then, it is a simple matter to show that, acting on conserved currents,

$$D^{GF}_{\mu\nu\rho\sigma} = D^{m}_{\mu\nu\rho\sigma} + \frac{c_1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} \tag{3.22}$$

JHEP03(2005)002

which means that there is an extra admixture of spin $s = 0$ in the massless case. The conventional normalization corresponds to

$$c_2 = \frac{4}{k^2} \tag{3.23}$$

and yields

$$K^{GF}_{\mu\nu\rho\sigma} = \frac{k^2}{8}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})\,. \tag{3.24}$$

This corresponds to the massless Fierz-Pauli lagrangian with the harmonic gauge condition

$$L_{GF} = \frac{1}{2}\left(\partial_\nu h_\mu{}^\nu - \frac{1}{2}\partial_\mu h\right)^2 \tag{3.25}$$

that is

$$L_0 = \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 - \frac{1}{8}(\partial_\mu h)^2\,. \tag{3.26}$$

3.2 Unimodular lagrangians

If we implement the restricted gauge symmetry only, a simpler lagrangian exists:

$$L_u = \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 - \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\alpha h^{\mu\beta} \tag{3.27}$$

although the full Fierz-Pauli lagrangian L_{FP} is obviously still invariant under the restricted symmetry. This is exactly the same thing that would have been gotten by putting $h = 0$ in the Fierz-Pauli lagrangian, that is

$$(K_u)_{\mu\nu\rho\sigma} = \frac{k^2}{8}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) + -\frac{1}{8}(k_\mu k_\rho\eta_{\nu\sigma} + k_\rho k_\nu\eta_{\mu\sigma} + k_\sigma k_\nu\eta_{\mu\rho} + k_\sigma k_\mu\eta_{\nu\rho})\,. \tag{3.28}$$

Let us now construct a massive unimodular theory. In order to do that, we postulate the most general mass term, say

$$-\frac{m^2}{8}(2h_{\mu\nu}h^{\mu\nu} - rh^2) \tag{3.29}$$

where r is an arbitrary constant (which for the full Fierz-Pauli theory happens to take the value $r = 2$). The posited full kinetic operator is then

$$\begin{aligned}(K^m_u)_{\mu\nu\rho\sigma} &= \frac{k^2 - m^2}{8}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) + r\frac{m^2}{8}\eta_{\mu\nu}\eta_{\rho\sigma} - \\ &\quad -\frac{1}{8}(k_\mu k_\rho\eta_{\nu\sigma} + k_\rho k_\nu\eta_{\mu\sigma} + k_\sigma k_\nu\eta_{\mu\rho} + k_\sigma k_\mu\eta_{\nu\rho})\,.\end{aligned} \tag{3.30}$$

The corresponding equation of motion is:

$$(K^m_u h)_{\mu\nu} = \frac{k^2 - m^2}{4}h_{\mu\nu} + \frac{rm^2}{8}h\eta_{\mu\nu} - \frac{1}{4}(k_\mu k_\rho h_\nu{}^\rho + k_\nu k_\rho h_\mu{}^\rho)\,. \tag{3.31}$$

Computing again the transverse part of the equation of motion:

$$(K^m_u.h)_{\mu\nu} = \frac{k^2 - m^2}{4}h_{\mu\nu} + r\frac{m^2}{8}h\eta_{\mu\nu} - \frac{1}{4}(k_\mu k^\rho h_{\nu\rho} + k_\nu k^\rho h_{\mu\rho}) \tag{3.32}$$

$$k^\mu k^\nu(K^m_u)_{\mu\nu\rho\sigma}h^{\rho\sigma} = -(k^2 + m^2)k_\rho k_\sigma h^{\rho\sigma} + r\frac{m^2k^2}{2}h = 0 \tag{3.33}$$

and the trace:

$$\eta^{\mu\nu}(K_u^m)_{\mu\nu\rho\sigma}h^{\rho\sigma} = \left(k^2 - m^2 + \frac{n}{2}rm^2\right) h - 2k_\rho k_\sigma h^{\rho\sigma} = 0\,. \tag{3.34}$$

This two conditions enforce

$$h = k_\rho k_\sigma h^{\rho\sigma} = 0 \tag{3.35}$$

as long as $r > 0$. Even when $r = 0$ they do enforce full transversality, although tracelessness is then only guaranteed off shell

$$(k^2 - m^2)h = 0\,. \tag{3.36}$$

The conclusion of this analysis is that the unimodular theory becomes massive with a mass term of the Fierz-Pauli type.

3.3 Propagators

Logically, our attention should now turn to a discussion of the unimodular massive propagator. The fact is that, for the minimal model (3.27), supplemented by a mass term such as the one in (3.29), there is no propagator, because this lagrangian is singular. This is perhaps somewhat of a surprise, because there is no known gauge symmetry when the mass is nonvanishing, but it is nevertheless true. Actually, the situation is as follows: there is a particular mode, proportional to

$$(\eta_u^T)_{\rho\sigma} \equiv \left(k^2 + m^2 - r\frac{m^2}{2}\right)\eta_{\rho\sigma} - \left(k^2 + m^2 - r\frac{n}{2}m^2\right)\frac{k_\rho k_\sigma}{k^2} \tag{3.37}$$

such that

$$\left(K_u^m \eta_u^T\right)_{\mu\nu} \tag{3.38}$$

is transverse, i.e.

$$\left(K_u^m \eta_u^T\right)_{\mu\nu} k^\mu = 0\,. \tag{3.39}$$

Although this is not a zero mode *sensu stricto*, it is enough to make the lagrangian singular. The situation is somewhat strange. Nevertheless, we already know, because of the argument of the polarizations at the beginning of the present section, that the correct lagrangian for massive spin 2 is the Fierz-Pauli one, (3.9). On the other hand, we know that the model (3.27), (3.29) is the minimal one which can be extended to *exactly* the Fierz-Pauli one while keeping only the restricted gauge symmetry in the massless case.

While it would be interesting to further study the minimal theory, we shall therefore confine our attention from now on to the Fierz-Pauli lagrangian.

3.4 Gauge fixing

The harmonic gauge is not reachable in the massless unimodular theory, because the equation

$$\partial_\mu h^{\mu\nu}_{(0)} - \frac{1}{2}\partial^\nu h_{(0)} + \Box\xi^\nu = 0 \tag{3.40}$$

has got the integrability condition

$$2\partial_\alpha\partial_\beta h^{\alpha\beta}_{(0)} = \Box h_{(0)}\,. \tag{3.41}$$

JHEP03(2005)002

Incidentally, the same holds true for any covariant linear gauge which is also linear in derivatives, that is, of the form:

$$M_{\alpha\beta\gamma}h^{\beta\gamma} = 0 \tag{3.42}$$

with

$$M_{\alpha\beta\gamma} = c_1\eta_{\alpha\beta}k_\gamma + c_2\eta_{\alpha\gamma}k_\beta + c_3\eta_{\beta\gamma}k_\alpha\,. \tag{3.43}$$

The equations of motion before gauge fixing are

$$\Box h_{\alpha\beta} = \partial_\beta\partial^\sigma h_{\alpha\sigma} + \partial_\alpha\partial^\sigma h_{\beta\sigma} \tag{3.44}$$

which lead to

$$\Box h = 2\partial_\alpha\partial_\beta h^{\alpha\beta} \tag{3.45}$$

so that on shell the equations of motion are equivalent to the Fierz-Pauli ones, and the harmonic gauge is possible. This is true in spite of the fact that the unimodular wave operator is neither transverse ($k^\mu(K_u)_{\mu\nu\rho\sigma} \neq 0$) nor traceless ($\eta^{\mu\nu}(K_u)_{\mu\nu\rho\sigma} \neq 0$).

Indeed the two extra terms that appear in the Fierz-Pauli equations of motion are

$$-\frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \frac{1}{4}\eta_{\mu\nu}\Box h \tag{3.46}$$

which vanish due to (3.45), which still holds here. We knew already that much, because we have proven the on-shell equivalence of the two theories at the nonlinear level in the introduction.

When we want to define the quantum theory, by means of a path integral, for example, we could impose in the Fierz-Pauli theory the noncovariant gauge conditions

$$\begin{aligned} h &= 0 \\ \partial_\mu h^{\mu i} &= 0 \end{aligned} \tag{3.47}$$

whereas in the minimal unimodular theory we could impose

$$\partial_\mu h^{\mu i} = 0 \tag{3.48}$$

(we can consider that only the ξ^i are independent, whereas $\xi^0 = \int^{x^0}\vec{\nabla}\vec{\xi}.$) But the trace h is not necessarily zero, so that the action off shell differs in general from the Fierz-Pauli value, and it is easy to convince oneself that there is not a gauge in which the two actions coincide. This is enough to show that the two theories are different at the quantum level, because the action measures the quantum phase associated to each path.

All this remains true if we choose the full Fierz-Pauli lagrangian L_{FP} as our starting point before restricted gauge fixing. In that case, the unimodular path integral is

$$Z_u(J) = \int \mathcal{D}h\, e^{iS_{FP}+i\int d^n x \frac{1}{2\alpha}(\partial_\mu h^{\mu i})^2 + i\int d^n x J_{\alpha\beta}h^{\alpha\beta}} \tag{3.49}$$

whereas with the full symmetry the closest we can get is

$$Z(J) = \int \mathcal{D}h\, \delta(h)\, e^{iS_{FP}+i\int d^n x \frac{1}{2\alpha}(\partial_\mu h^{\mu i})^2 + i\int d^n x J_{\alpha\beta}h^{\alpha\beta}} \tag{3.50}$$

We are not worrying about ghosts because we remain in the abelian approximation. The

JHEP03(2005)002

conclusion of the present analysis is that even in the non-interacting case the two theories are different, and this in spite of the fact that the starting lagrangian is the same, owing to the different gauge fixing.

4. Full covariant unimodular lagrangians.

One of the most interesting full covariant lagrangian (i.e. diffeomorphism invariant) versions of the same traceless equations of motion is the one proposed in [4], in terms of a $(n-1)$-differential form, A_{n-1} and a scalar field, $\lambda(x)$, namely

$$S = \frac{1}{2\kappa^2}\left(\int d^n x\sqrt{|g|}(R+2\lambda) - 2\int A\wedge d\lambda\right) \tag{4.1}$$

The equations of motion for the metric tensor are Einstein's equations with a cosmological scalar function:

$$\frac{\delta S}{\delta g^{\alpha\beta}} = R_{\alpha\beta} - \frac{1}{2}(R+2\lambda(x))g_{\alpha\beta}\,. \tag{4.2}$$

The equation of motion for the field $\lambda(x)$ is

$$\frac{\delta S}{\delta\lambda} = \sqrt{|g|}d^n x - dA \tag{4.3}$$

and finally, the equation of motion for the A-form is:

$$\frac{\delta S}{\delta A} = d\lambda\,. \tag{4.4}$$

There is an extra symmetry of the theory, namely

$$\delta A = d\Omega \tag{4.5}$$

which is the one that plays the rôle of the unimodular transformation, just by defining

$$\epsilon^{\mu} \equiv \frac{1}{2}\delta^{\mu\alpha_1\ldots\alpha_{n-1}}\partial_{\alpha_1}\Omega_{a_2\ldots\alpha_{n-1}} \tag{4.6}$$

which indeed obeys

$$\partial_\mu\epsilon^\mu = 0\,. \tag{4.7}$$

In this case it is even more clear than in the examples above that the full quantum theory is different in principle from the one stemming from Einstein-Hilbert's lagrangian, at least insofar as both of them possess different action principles.

5. Conclusions

It is well-known that the coupling of matter to the Fierz-Pauli lagrangian is inconsistent unless the full nonlinear general relativity is reconstructed (cf. for example, the general review in [14], where further references can be found). From this point of view, it seems that the natural nonlinear completion of the unimodular theory is the Einstein-Hilbert lagrangian, dressed with arbitrary functions of the determinant of the metric.

A straightforward calculation would then predict different perturbative amplitudes for both theories. Outside the realm of perturbation theory, all speculation is possible. An interesting question is, for example, how this restricted unimodular symmetry can (if at all) be implemented in a scheme such as the one called *loop quantum gravity*.

JHEP03(2005)002

Given the fact that the equations of motion are identical, and, as we have shown, this property remains true when higher order (in the curvature) corrections are considered, it is clear that the identification of the low energy limit of string theories as general relativity is premature; it could easily be the unimodular theory we are considering. Reliable computation of stringy off-shell correlators could, of course, be decisive in this respect.

Let us consider (cf. for example, [15]) the free energy in the presence of arbitrary conserved sources. This quantity is an exceedingly useful one to consider, because in summarizes in a very simple way the physical content of the theory. We shall assume two spatially disconnected sources: $T_{\alpha\beta} \equiv (T_1)_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{x}_1) + (T_2)_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{x}_2)$, with

$$\partial_\alpha (T_1)^{\alpha\beta} = \partial_\alpha (T_2)^{\alpha\beta} = 0\,. \tag{5.1}$$

Keeping only the term bilinear in the sources, assumed to act for a total time interval $\int dx^0 \equiv T$, one easily gets:

$$W = -\frac{2}{3}T\int d^3k \frac{1}{\vec{k}^2+m^2} e^{i\vec{k}(\vec{x}-\vec{y})}E_{12}\,. \tag{5.2}$$

Starting with the massive Fierz-Pauli theory, the answer stemming from (3.5) is

$$E_{12} = (\mathrm{tr}\, T_1\, \mathrm{tr}\, T_2 - 3\, \mathrm{tr}\, T_1 T_2)\,. \tag{5.3}$$

In the massless case, the Fierz-Pauli interacion energy in the harmonic gauge is proportional instead to

$$E_{12} \equiv \frac{1}{2}\left(2\, \mathrm{tr}\, T_1 T_2 - (\mathrm{tr}\ T_1)(\mathrm{tr}\ T_2)\right). \tag{5.4}$$

Even forgetting about the coefficients, there is a mismatch of $3/2$ in the term tr T_1T_2; this is the famous van Dam-Veltman discontinuity ([16]), which indicates that there is some sort of non smoothness in the massless limit.

In full[2] detail:

$$\begin{aligned} E_{12} \;=\; & \frac{1}{2}T_1^{00}(T_2^{00}+T_2^{11}+T_2^{22}+T_2^{33}) + \frac{1}{2}T_1^{11}(T_2^{00}+T_2^{11}-T_2^{22}-T_2^{33}) + \\ & +\frac{1}{2}T_1^{22}(T_2^{00}-T_2^{11}+T_2^{22}-T_2^{33}) + \frac{1}{2}T_1^{33}(T_2^{00}-T_2^{11}-T_2^{22}+T_2^{33}) + \\ & +2\left(T_1^{12}T_2^{12}+T_1^{13}T_2^{13}+T_1^{23}T_2^{23}-T_1^{01}T_2^{01}-T_1^{02}T_2^{02}-T_1^{03}T_2^{03}\right). \end{aligned} \tag{5.7}$$

[2]The resulting expression can be further simplified using current conservation:

$$\begin{aligned} T^{00} &= \frac{\kappa}{\omega}T^{03} = \frac{\kappa^2}{\omega^2}T^{33} \\ T^{0i} &= \frac{\kappa}{\omega}T^{3i} \end{aligned} \tag{5.5}$$

getting

$$\begin{aligned} E_{12} \;=\; & \frac{1}{2}\left(T_1^{11}-T_1^{22}\right)\left(T_2^{11}-T_2^{22}\right) + 2T_1^{12}T_2^{12} + \frac{m^2}{2\omega^2}\left(T_1^{11}+T_1^{22}\right)T_2^{33} + 2\frac{m^2}{\omega^2}\left(T_1^{13}T_2^{13}+T_1^{23}T_2^{23}\right) + \\ & +\frac{1}{2}T_1^{33}\left(\frac{m^4}{\omega^4}T_2^{33} - \frac{m^2}{\omega^2}\left(T_2^{11}+T_2^{22}\right)\right). \end{aligned} \tag{5.6}$$

This clearly shows that in the massless limit only the two polarizations in (3.18) contribute (cf. [15][17]) to this physical observable.

In order to identify possible off-shell intermediate states in the massless Fierz-Pauli theory, it is useful to transform the expression (5.7) (that is, before using current conservation) into the suggestive form proposed by Dicus and Willenbrock [17].

$$\begin{aligned}E_{12} = &\frac{1}{2}(T_1^{11} - T_1^{22})(T_2^{11} - T_2^{22}) + 2T_1^{12}T_2^{12} + 2\left(T_1^{13}T_2^{13} + T_1^{23}T_2^{23} - T_1^{01}T_2^{01} - T_1^{02}T_2^{02}\right) \times\\ &\times\frac{1}{6}[2(T_1^{00} - T_1^{33}) + T_1^{11} + T_1^{22}][2(T_2^{00} - T_2^{33}) + T_2^{11} + T_2^{22}] -\\ &-\frac{1}{6}[-T_1^{00} + T_1^{11} + T_1^{22} + T_1^{33}][-T_2^{00} + T_2^{11} + T_2^{22} + T_2^{33}]\end{aligned} \tag{5.8}$$

This can be easily checked: in order for the coefficient of T_1^{00} in (5.8) to be the same as the one in (5.7) we have to add a term $T_1^{00}T_2^{33}$, and also if we want the coefficient of T_1^{33} in (5.8) to be the same as in (5.7) we have to add another term $T_1^{33}T_2^{00}$. But in order for the coefficients of T_1^{03} to match, we have to add $-2T_1^{03}T_2^{03}$, which exactly cancel owing to the conservation of the energy momentum tensor.

This expansion can be spelled down physically as follows. Let us introduce a real basis of polarizations in the generic case as

$$\begin{aligned}\epsilon_3 &= e_{(0)} \otimes e_{(0)} - e_{(1)} \otimes e_{(1)}\\ \epsilon_4 &= e_{(0)} \otimes e_{(1)} + e_{(1)} \otimes e_{(0)}\\ \epsilon_5 &= e_{(0)} \otimes e_{(2)} + e_{(2)} \otimes e_{(0)}\end{aligned} \tag{5.9}$$

Then the second line of (5.8) is proportional to:

$$T_1^{\mu\nu}(2\epsilon_4 + \epsilon_5)_{\mu\nu}(2\epsilon_4 + \epsilon_5)_{\rho\sigma}T_2^{\rho\sigma} \tag{5.10}$$

and the third one to

$$T_1^{\mu\nu}(\epsilon_2 + 2e_3)_{\mu\nu}(\epsilon_2 + 2e_3)_{\rho\sigma}T_2^{\rho\sigma} \tag{5.11}$$

whereas the last one is a spin zero contribution

$$T_1^{\mu\nu}(\epsilon_1 + \epsilon_2 + e_3)_{\mu\nu}(\epsilon_1 + \epsilon_2 + e_3)_{\rho\sigma}T_2^{\rho\sigma} \tag{5.12}$$

which obviously does not correspond to spin two, but is nevertheless neccessary to cancel the contribution of the unphysical polarizations in the massless case. So that not only are off-shell spin zero components allowed by the theory as intermediate states, but as has been pointed out by Dicus and Willenbrok, they are actually neccessary for consistency. The appearance of these components was first pointed out in [18].

Coming back to our main theme, a natural question is how can we experimentally discriminate between both theories? There is an easy answer, namely that graviton scattering amplitudes are expected to be different in detail. But unfortunately, graviton scattering data do not abound.

A most interesting, and perhaps feasible experiment would be to weigh the vacuum energy, i.e. Casimir energy. Indeed, under the restricted variations in (2.5) which we have labelled $\delta^t g_{\mu\nu}$, the vacuum energy does not affect the equations of motion.

A related point is the following. Granting that the two Einstein theories are indeed different at the quantum level, the most important physical question is whether this improves or otherwise reformulates in some way the problem of the cosmological constant. Interesting suggestions in this direction have been made by [19] and [20], although no compelling model exists yet.

Acknowledgments

I have benefited from discussions with E. Gabrielli, Tomás Ortín and (long time ago, but still relevant for this paper) with Tini Veltman. This work has been partially supported by the European Commission (HPRN-CT-200-00148) and by FPA2003-04597 (DGI del MCyT, Spain).

References

[1] W.G. Unruh, *A unimodular theory of canonical quantum gravity*, *Phys. Rev.* **D 40** (1989) 1048.

[2] A. Einstein, Siz. Preuss. Acad. Scis. 1919, english translation in *The principle of relativity*, A. Einstein et al. eds., Dover.

[3] A. Daughton, J. Louko and R.D. Sorkin, *Instantons and unitarity in quantum cosmology with fixed four-volume*, *Phys. Rev.* **D 58** (1998) 084008 [gr-qc/9805101].

[4] M. Henneaux and C. Teitelboim, *The cosmological constant and general covariance*, *Phys. Lett.* **B 222** (1989) 195.

[5] M. Kreuzer, *Gauge theory of volume preserving diffeomorphisms*, *Class. and Quant. Grav.* **7** (1990) 1303.

[6] Y.J. Ng and H. van Dam, *A small but nonzero cosmological constant*, *Int. J. Mod. Phys.* **D 10** (2001) 49 [hep-th/9911102].

[7] J.J. van der Bij, H. van Dam and Y.J. Ng, *Theory of gravity and the cosmological term: the little group viewpoint,Physica* **116A** (1982) 307.

[8] K.-I. Sato, *New formulation of massive spin two field theory and solution of vdvz discontinuity problem*, hep-th/0501042.

[9] T. Padmanabhan, *From gravitons to gravity: myths and reality*, gr-qc/0409089.

[10] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, Proc.Roy.Soc.Lond1731939211.

[11] M.J.G. Veltman, *Quantum theory of gravitation*, in *Methods in field theory*, Les Houches, 1975.

[12] T. Ortin, *Gravity and strings*, Cambridge University press, Cambridge 2004.

[13] P.Van Nieuwenhuizen, *On ghost-free tensor lagrangians and linearized gravitation*, *Nucl. Phys.* **B 60** (1973) 478.

[14] E. Alvarez, *Quantum gravity: a pedagogical introduction to some recent results*, *Rev. Mod. Phys.* **61** (1989) 561.

[15] A. Zee, *Quantum field theory in a nutshell*, Princeton University press, Princeton 2003.

[16] H. van Dam and M.J.G. Veltman, *Massive and massless Yang-Mills and gravitational fields*, *Nucl. Phys.* **B 22** (1970) 397;
V.I. Zakharov, *JETP Lett.* **12** (1970) 312;
A.I. Vainshtein, *To the problem of nonvanishing gravitation mass*, *Phys. Lett.* **B 39** (1972) 393.

[17] D. Dicus and S. Willenbrock, *Angular momentum content of a virtual graviton*, hep-ph/0409316.

[18] A. Datta, E. Gabrielli and B. Mele, *Violation of angular momentum selection rules in quantum gravity*, *Phys. Lett.* **B 579** (2004) 189 [hep-ph/0309232]; *More on violation of angular-momentum selection rules in quantum gravity*, hep-ph/0406149;
T. Han and S. Willenbrock, *Scale of quantum gravity*, hep-ph/0404182.

[19] A.A. Tseytlin, *Duality symmetric string theory and the cosmological constant problem*, *Phys. Rev. Lett.* **66** (1991) 545.

[20] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and G. Gabadadze, *Non-local modification of gravity and the cosmological constant problem*, hep-th/0209227.